Introduction

1) let X sm proj $/ K \subset \mathbb{C}$

$\rightsquigarrow$ $X(\mathbb{C})$ is a cpt cplx mfld.

$\rightarrow$ $H^i(X(\mathbb{C}), \mathbb{Z})$ singular coh

and have

$$H^i(X(\mathbb{C}), \mathbb{C}) \underset{\substack{\uparrow \\ \mathbb{C} \to \Omega^\bullet_{X(\mathbb{C})/\mathbb{C}} \\ \text{resolution}}}{\cong} H^i_{dR}(X(\mathbb{C})/\mathbb{C}) \underset{\substack{\uparrow \\ \text{Serre} \\ \text{(GAGA)}}}{\cong} H^i_{dR}(X/K) \otimes_K \mathbb{C}$$

$$\|$$

$$H^i(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

$\rightsquigarrow$ rich structure: · get an $\mathbb{Q}$-vsp str on $H^i_{dR}(X/K) \otimes_K \mathbb{C}$

· get Hodge filtr on $H^i(X(\mathbb{C}), \mathbb{C})$

· Topology $\leftrightarrow$ Alg Geom.

$\rightsquigarrow$ Hodge Theory.

Ex: X Enriques surface, i.e.

X sm proj $/K$ connected, $b_2 = \dim_{\mathbb{Q}} H^2(X(\mathbb{C}), \mathbb{Q}) = 10$

$\omega_X^{\otimes 2} \cong \mathcal{O}_X$ (where $\omega_X = \Omega^2_{X/K}$)

$\hookrightarrow$ defines fin ét $Y \longrightarrow X$ $\mathbb{Z}/2\mathbb{Z}$-Torsor.

$\uparrow$ K3 $\rightarrow$ has $\pi^1(Y(\mathbb{C})) = 0$

$\Rightarrow \pi_1(X(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow$ only interesting information in

$$H^1(X(\mathbb{C}), \underset{\uparrow \text{ab gp}}{A}) = \mathrm{Hom}(\pi_1(X(\mathbb{C})), A)$$

when A is 2-torsion!

→ study

$$\begin{tikzcd}
H^i(X(\mathbb{C})^{an}, \mathbb{Z}/p^n) \arrow[d, equal, "\text{Artin}"] & \text{for a prime } p \\
H^i_{\acute{e}t}(X_{\mathbb{C}}, \mathbb{Z}/p^n) \arrow[d, equal] & \\
H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}/p^n) & \\
\circlearrowleft & \\
\mathrm{Gal}(\bar{K}/K) &
\end{tikzcd}$$

is there any extra de Rham-like information ?

p-adic Hodge theory

Assume $K/\mathbb{Q}_p$ fin ext with ring of integers $\mathcal{O}_K$ and residue field $k = \mathcal{O}_K/m_K \supset \mathbb{F}_p$

assume $\exists$ $\mathfrak{X}$ sm proj $/\mathcal{O}_K$ s.t. $X = \mathfrak{X} \otimes_{\mathcal{O}_K} K$

$$\Rightarrow \left(\varprojlim H^i_{et}(X_{\bar{K}}, \mathbb{Z}/p^n)\right) \otimes_{\mathbb{Z}_p} B_{cris}^+ \simeq H^i_{crys}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} B_{cris}^+$$

$\uparrow$ $W(k)[1/p]$-alg

(in part $\mathbb{Q}_p$-alg)

with filtration, Gal-action, Frobenius

$$\rightarrow \quad H^i_{et}(X_{\bar{K}}, \mathbb{Q}_p) = \left(H^i_{crys}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} B_{cris}^+\right)^{Fil^0, Frob=1}$$

(Fontaine, Faltings, Tsuji, ...)

But no torsion information!

Note however

$$H^i_{crys}(\mathfrak{X}_k/W(k)) = H^i(R\Gamma_{crys}(\mathfrak{X}_k/W(k)))$$

and $R\Gamma_{crys}(\mathfrak{X}_k/W(k)) \otimes^L_{W(k)} k = R\Gamma_{dR}(\mathfrak{X}_k/k)$

$\uparrow$

cx of k-v sp's

Q: Is there a relation between

$$H^i(X(\mathbb{C}), \mathbb{Z}/p\mathbb{Z}) = H^i_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$$

and

$$H^i_{dR}(\mathcal{X}_k/k) \qquad ?$$

No easy relation:

- not equal and $H^i(X(\mathbb{C}), \mathbb{Z}/p\mathbb{Z})$ does not only depend on special fiber $\mathcal{X}_k$

  indeed $\exists\ \mathcal{X}, \mathcal{Y}$ sm proj 3-folds / $\mathbb{Z}_2$

  s.t. $\mathcal{X}_{\mathbb{F}_2} \simeq \mathcal{Y}_{\mathbb{F}_2}$ but $H^1(\mathcal{X}(\mathbb{C}), \mathbb{F}_2) = \mathbb{F}_2^2$

  $H^1(\mathcal{Y}(\mathbb{C}), \mathbb{F}_2) = \mathbb{F}_2^3$

- There is no natural inclusion of the gps.

But

Thm (BMS, BS)

$$\dim_{\mathbb{F}_p} H^i(X(\mathbb{C}), \mathbb{F}_p) \leq \dim_k H^i_{dR}(\mathcal{X}_k/k)$$

Cor: If $\dim_k H^i_{dR}(\mathcal{X}_k/k) = \dim_K H^i(\mathcal{X}_K/K) \Rightarrow H^{i+1}(X(\mathbb{C}), \mathbb{Z})$ has no p-torsion.

Proof of the Thm use Prismatic coh.

→ <u>sketch of defn</u>

- let A be a p-torsion-free ring

  it is a <u>δ-ring</u> if

  $\exists$ $\delta : A \to A$ map s.t.

  $$\phi(a) := a^p + p\,\delta(a)$$

  defines a ring hom $\phi : A \to A$

  (→ Frobenius lift)

(→ 2nd talk (Picard)
3rd talk (Dooseung))

- A <u>bounded prism</u> is a pair (A, I)

where $A = (A, \delta)$ is a δ-ring

$I \subset A$ is an ideal

s.t.
- I is locally gen by a non-zero div
- A is derived (p, I)-complete (→ talk 5 on May 10)
- $p \in I + \phi(I)A$
- A/I has bounded p-torsion.

(→ talk 6 (May 17) + talk 7 (May 24))

Examples

- $A = \mathbb{Z}_p$, $I = (p)$

  with $\phi = id$ ($\Rightarrow \delta(x) = \frac{x - x^p}{p}$)

⌈ For more general R = perf pos char ⌉

$A = W(R)$, $I = (p)$

$\phi = W(\sigma)$ $\quad \sigma : R \to R$, $x \mapsto x^p$ ⌋

⌊

- $C / \mathbb{Q}_p$ complete alg closed

  (non-arch analog of $\mathbb{C}$)

$\mathcal{O}_C$ ring of integers

$\mathcal{O}_C^\flat := \varprojlim_{\phi} \mathcal{O}_C / p \ni (a_1, a_2, a_3, \ldots\ldots)$ (↳ talk 3)

$a_n = a_{n+1}^p$

perfect ring of char p

Witt vectors

$\rightarrow \exists$ surj ring hom $\theta : A_{inf}(\mathcal{O}_C) := W(\mathcal{O}_C^\flat) \twoheadrightarrow \mathcal{O}_C$

$I := \ker(\theta)$

$\Rightarrow (A_{inf}(\mathcal{O}_C), I)$ is a perfect prism

↳ ϕ is bij.

(all perfect prisms are of this form w/ $\mathcal{O}_C \leftrightarrow$ perfectoid)

- $\left(\mathbb{Z}_p[[q-1]], \underset{\overset{\shortparallel}{\frac{q^p-1}{q-1}}}{[p]_q}\right)$ with $\delta(q)=0$ $\left(\Rightarrow \phi(1-q)=1-q^p\right)$

is a prism $\longrightarrow$ q-dR-coh
q-crystalline-coh

(see talk 10 (June 28)
talk 11 (July 5)
talk 12 (July 12)

- Prismatic site $/(A, I)$ $\leftarrow$ base prism

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let X smooth $/ A/I$

$(X/A)_{\Delta}$ has objects

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[d] & \operatorname{Spec} B/IB \arrow[l, hook'] \arrow[r] \arrow[d] & X \arrow[dl] \\
\operatorname{Spec} A & \operatorname{Spec} A/I \arrow[l, hook'] &
\end{tikzcd}$$

with (B, IB) prism $/(A, I)$

with "flat topology"

Then $\mathcal{O}_{\Delta}(\operatorname{Spec} B \hookleftarrow \operatorname{Spec} B/IB \to X) := B$ define a sheaf on $(X/A)_{\Delta}$

$\rightsquigarrow R\Gamma_{\Delta}(X/A) := R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta})$ prismatic cohomology

Thm (Bhatt-Scholze)

(A, I) bounded prism. X smooth over A/I

(1) Crystalline Comparison

$I = (p)$ [PD-ideal in A]

$$\Rightarrow \phi_A^* R\Gamma_{\Delta}(X/A) \cong R\Gamma_{crys}(X/A)$$

(2) Hodge-Tate Comparison

$X = \operatorname{Spec} R$

$$\Omega^i_{R/(A/I)}\{-i\} \cong H^i\left(R\Gamma_{\Delta}(X/A) \overset{L}{\underset{A}{\otimes}} A/I\right)$$

(3) Étale Comparison

(A, I) perfect prism

$$R\Gamma_{ét}(X_{\mathbb{Q}_p}, \mathbb{Z}/p^n\mathbb{Z}) \cong \left(R\Gamma_{\Delta}(X/A)/p^n[\tfrac{1}{I}]\right)^{\phi=1}$$

(1), (2) → Talk 8 June 14
9 June 21

sketch of pf of : $\Big\| \dim_{\mathbb{F}_p} H^i_{et}(\mathfrak{X}_{\bar{K}}, \mathbb{Z}/p\mathbb{Z}) \leq \dim_k H^i_{dR}(\mathfrak{X}_k/k)$:

for $\mathfrak{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ sm proj

Take $C \supset K \supset \mathbb{Q}_p$, wlog $\mathfrak{X}/\mathcal{O}_C$

$\uparrow$ complete alg closed.

$A = A_{inf}(\mathcal{O}_C)$, $\ker(\theta : A_{inf}(\mathcal{O}_C) \longrightarrow \mathcal{O}_C) = I = (d)$

$R\Gamma_A(\mathfrak{X}) := R\Gamma_{\Delta}(\mathfrak{X}/A)$ $\quad (\xrightarrow[HT]{}$ perfect cx of A-mod.)

$V := A/p = \mathcal{O}_C^\flat$ perfect val ring

$\operatorname{Frac}(V) = C^\flat = V[1/d]$

$k = \mathcal{O}_C^\flat / \mathfrak{m}$ res field

$$\Longrightarrow \quad \dim_{C^\flat} H^i(R\Gamma_A(\mathfrak{X}) \otimes^L_A C^\flat) \leq \dim_k H^i(R\Gamma_A(\mathfrak{X}) \otimes^L_A k)$$

$\|$ crys comp

$\dim_k H^i_{dR}(\mathfrak{X}_k/k)$

$\|$

$\dim_{\mathbb{F}_p} H^i_{et}(\mathfrak{X}_C, \mathbb{F}_p)$

étale comp Thm